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Geometric inequalities for spacelike hypersurfaces in the Minkowski spacetime[☆]

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Abstract

We derive a linear isoperimetric inequality and some geometric inequalities for properly located compact achronal spacelike hypersurfaces via a Minkowski-type integral formula in the Minkowski spacetime.

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1. Introduction

Let M be a compact achronal spacelike hypersurface in the Minkowski $(n+1)$-spacetime $\mathbb{L}^{n+1}$ contained in the chronological future $I^+(O)$ of the origin O in $\mathbb{L}^{n+1}$. Let $\mathcal{K}_M$ denote the *cone* of M (with respect to O) given by

$$\mathcal{K}_M = \{\lambda p \in \mathbb{L}^{n+1} : p \in M, 0 \leq \lambda \leq 1\},$$

and $\Omega = \Omega(M)$ the *normalized hypersurface* of M given by

$$\Omega = \Omega(M) = \left\{ \frac{x}{\|x\|} \in \mathbb{L}^{n+1} : x \in M \right\},$$

where $\|x\| > 0$ is the norm of x. Note that $\Omega(M)$ is also a compact achronal spacelike hypersurface contained in $I^+(O)$. By $V(B)$, we denote the $(n+1)$-dimensional volume of $B \subset \mathbb{L}^{n+1}$ and by $A(B)$ the n-dimensional (Lorentzian) volume of $B \subset \mathbb{L}^{n+1}$. Recently,

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Bahn and Ehrlich [3] obtained the following isoperimetric inequality for such a spacelike hypersurface in the Minkowski spacetime $\mathbb{L}^{n+1}$ as a consequence of a Brunn–Minkowski type inequality in the Minkowski spacetime $\mathbb{L}^{n+1}$.

Theorem 1. *Let M be a compact achronal spacelike hypersurface in $\mathbb{L}^{n+1}$ contained in $I^+(O)$ with the* (*Lorentzian*) *distance* $\mathrm{d}(O, M) = t^*$. *Then*

$$A^{n+1}(M) \le (n+1)^{n+1} V(\mathcal{K}_\Omega) V^n(\mathcal{K}_M),$$

where equality holds only when M is contained in the (*upper*) *hyperbolic space $\mathbb{H}^n(t^*)$ of radius t^*.*

This isoperimetric inequality is a Lorentzian version of the isoperimetric inequality for convex cones of Lions and Pacella [7].

In this paper, we consider geometric inequalities for a compact spacelike hypersurface M in the Minkowski $(n+1)$-spacetime $\mathbb{L}^{n+1}$ involving the mean curvature H of M. Notice that the quantities $V(\mathcal{K}_\Omega)$ and $V(\mathcal{K}_M)$ in Theorem 1 depend on the location of M in $\mathbb{L}^{n+1}$. So, for our purpose, we restrict our attention to a compact spacelike hypersurface $M \subset \mathbb{L}^{n+1}$ such that M is contained in the chronological future $I^+(O)$ of the origin O in $\mathbb{L}^{n+1}$ and

$$\int_{\partial M} \langle P, \nu \rangle \, \mathrm{d}S \ge 0, \tag{1}$$

where ν is the outward unit conormal vector field along ∂M and P the position vector field in $\mathbb{L}^{n+1}$. Such a compact spacelike hypersurface M will be said to be *properly located*. Note that "$\langle P, \nu \rangle \ge 0$" represents that the hyperbolic angle from ν to P is non-negative (cf. [2,4]). For some properly located compact spacelike hypersurfaces $M \subset \mathbb{L}^{n+1}$, we derive a linear isoperimetric inequality via a Minkowski-type integral formula (see Lemma 3). Finally, we prove the following geometric inequalities in Theorems 6 and 7.

Theorem 2. *Let M be a properly located compact achronal spacelike hypersurface in $\mathbb{L}^{n+1}$ with mean curvature $H \ge c > 0$. Then*

$$V(\mathcal{K}_M) \le \frac{1}{c^{n+1}} V(\mathcal{K}_\Omega), \qquad A(M) \le \frac{1}{c^n} A(\Omega)$$

with equality only when M is a subset of $\mathbb{H}^n(1/c)$.

Note that, for our convention, the (upper) hyperbolic space $\mathbb{H}^{n+1}(r)$ of radius $r > 0$ has positive constant mean curvature $1/r$.

For detailed account of the terminology relating to indefinite metric and causality, the reader may consult [5] or [8].

2. Minkowski-type integral formula

The Minkowski $(n+1)$-spacetime $\mathbb{L}^{n+1} (n \ge 2)$ is just the real vector space $\mathbb{R}^{n+1}$ with the scalar product g of index 1,

$$g(x, y) = \langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$$

for $x = (x_0, x_1, \dots, x_n)$, $y = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$. We will assume, as usual, $\mathbb{L}^{n+1}$ is time-oriented by the unit timelike vector $e_0 = (1, 0, \dots, 0)$ so that a non-spacelike tangent vector $v \in \mathbb{L}^{n+1}$ is said to be future-directed if $\langle e_0, v\rangle < 0$. We denote by $\mathbb{H}^n(r)$ the (*upper*) *hyperbolic n-space* of radius $r > 0$ in $\mathbb{L}^{n+1}$ given by

$$\mathbb{H}^n(r) = \{x = (x_0, x_1, \dots, x_n) \in \mathbb{L}^{n+1} : \langle x, x\rangle = -r^2, x_0 > 0\},$$

which is a complete spacelike hypersurface of in the Minkowski spacetime $\mathbb{L}^{n+1}$ with constant curvature $-1/r^2$.

Let M be a compact spacelike hypersurface in $\mathbb{L}^{n+1}$ oriented by the future-directed unit timelike normal vector field N. Note that every compact spacelike hypersurface M in $\mathbb{L}^{n+1}$ necessarily has a non-empty boundary ∂M (cf. [1]). So, for simplicity, we will always assume that M has smooth boundary ∂M. We will decompose the position vector field P, restricted on M, as

$$P = P^{\mathrm{T}} - \langle P, N\rangle N, \tag{2}$$

where P^{T} denotes the tangential part of P to M. Let $\bar{\nabla}$ and ∇ denote the Levi–Civita connections of $\mathbb{L}^{n+1}$ and M, respectively. Then, the Gauss and Weingarten formulas for M in $\mathbb{L}^{n+1}$ are written, respectively, as

$$\bar{\nabla}_X Y = \nabla_X Y - \langle S(X), Y\rangle N, \tag{3}$$

$$S(X) = -\bar{\nabla}_X N, \tag{4}$$

for all smooth tangent vector fields $X, Y \in \mathfrak{X}(M)$ on M, where S is the shape operator of M in $\mathbb{L}^{n+1}$ associated with N. The mean curvature H on M is given by

$$H = -\frac{1}{n}\,\mathrm{tr}(S). \tag{5}$$

Now, we can formulate a Lorentzian version of a Minkowski formula (cf. [6]) in the Minkowski spacetime as given in the following lemma.

Lemma 3. *Let M be a compact spacelike hypersurface with smooth boundary ∂M in the Minkowski spacetime $\mathbb{L}^{n+1}$. Then*

$$\frac{1}{n}\oint_{\partial M} \langle P, \nu\rangle\, \mathrm{d}S = A(M) + \int_M \langle P, N\rangle H\, \mathrm{d}A, \tag{6}$$

where $\mathrm{d}A$ *is the n-dimensional volume element on M,* $\mathrm{d}S$ *the induced* $(n-1)$*-dimensional volume element on* ∂M.

Proof. Let $\phi = \langle P, P\rangle$, where P is the position vector field in $\mathbb{L}^{n+1}$. Then, the gradient of ϕ on M is given by

$$\nabla\phi = 2P^{\mathrm{T}} = 2(P + \langle P, N\rangle N),$$

so the Hessian of ϕ on M is given as

$$\nabla^2\phi(X, Y) = \langle \nabla_X(\nabla\phi), Y\rangle = 2\langle X, Y\rangle - 2\langle P, N\rangle\langle S(X), Y\rangle,$$

where $X, Y \in \mathfrak{X}(M)$ by (3) and (4). Therefore, by (5), the Laplacian of ϕ on M is given by

$$\Delta\phi = 2n + 2n\langle P, N\rangle H.$$

Integrating this over M, by the divergence theorem, we have

$$\oint_{\partial M} \langle P^{\mathrm{T}}, \nu\rangle \,\mathrm{d}S = nA(M) + n\int_M \langle P, N\rangle H \,\mathrm{d}A,$$

which implies (6). □

Remark 4. Let M be a compact spacelike hypersurface in $\mathbb{L}^{n+1}$ with smooth boundary ∂M. Note that the condition (1) is equivalent to the weak super-harmonicity of $\phi = \langle P, P\rangle$, i.e.,

$$\int_M \Delta\phi \,\mathrm{d}A \geq 0.$$

Let $\Pi_p(P, \nu)$ be the subspace of tangent space $T_p(\mathbb{L}^{n+1})$ of $\mathbb{L}^{n+1}$ at $p \in \partial M$ spanned by P_p and ν_p. Let $\{u_0, u_1\}$ be an orthonormal basis of $\Pi_p(P, \nu)$, where u_0 is a future-directed unit timelike vector and $\langle u_1, \nu_p\rangle > 0$. Then, the scalar product of P_p and ν_p is given by

$$\langle P_p, \nu_p\rangle = \|P_p\| \sinh(\varphi(P_p) - \varphi(\nu_p)),$$

where $\varphi(v)$ is the *hyperbolic angle* of v with respect to $\{u_0, u_1\}$. Notice that the hyperbolic angle from ν_p to P_p, "$\varphi(P_p) - \varphi(\nu_p)$", is independent on the choice of the orthonormal basis $\{u_0, u_1\}$ in $\Pi_p(P, \nu)$ (cf. [2,4]). In particular, for a compact spacelike hypersurface in $\mathbb{L}^{n+1}$ with smooth boundary $\partial M \subset \mathbb{H}^n(r)$, the condition (1) means that the hyperbolic angle $\varphi(P)$ is greater than or equal to the hyperbolic angle $\varphi(\nu)$ in $\Pi(P, \nu)$ on average. For instance, consider the family of hyperbolic caps

$$M_c = \{x \in \mathbb{L}^{n+1} : \langle x, x\rangle = -c^2, 0 < x_0 \leq \sqrt{1 + c^2}\} \quad (0 < c < \infty).$$

Then all M_c are properly located, since $M_c \subset I^+(O)$ and $\langle P, \nu\rangle = 0$ on the boundary ∂M_c. Furthermore,

$$M_c - \vec{\varepsilon} = \{p - \vec{\varepsilon} \in \mathbb{L}^{n+1} : p \in M_c, \quad \vec{\varepsilon} = (\varepsilon, 0, \ldots, 0), \varepsilon \geq 0\}$$

is also properly located if $M_c - \vec{\varepsilon}$ is still contained in $I^+(O)$.

3. Geometric inequalities

We first derive a linear isoperimetric inequality for a properly located compact achronal spacelike hypersurface, which is given in the following lemma.

Lemma 5. *Let M be a properly located compact achronal spacelike hypersurface in* $\mathbb{L}^{n+1}$ *with mean curvature* $H \geq c > 0$. *Then*

$$V(\mathcal{K}_M) \leq \frac{1}{c(n+1)} A(M). \tag{7}$$

Proof. Since M is properly located, from the Minkowski formula (Lemma 3),

$$-\int_M \langle P, N\rangle H \,\mathrm{d}A \le A(M).$$

Note that P and N are future-directed, so $\langle P, N\rangle < 0$ on M. From the condition that $H \ge c > 0$, we have

$$-c\int_M \langle P, N\rangle \,\mathrm{d}A \le A(M).$$

Note that the $(n+1)$-dimensional volume of the cone $\mathcal{K}_M$ of M is

$$V(\mathcal{K}_M) = -\frac{1}{n+1}\int_M \langle P, N\rangle \,\mathrm{d}A.$$

Thus, we have

$$c(n+1)V(\mathcal{K}_M) \le A(M),$$

which is equivalent to (7). □

Theorem 6. *Let M be a properly located compact achronal spacelike hypersurface in* $\mathbb{L}^{n+1}$ *with mean curvature* $H \ge c > 0$. *Then*

$$V(\mathcal{K}_M) \le \frac{1}{c^{n+1}} V(\mathcal{K}_\Omega) \tag{8}$$

with equality only when M is a subset of $\mathbb{H}^n(1/c)$.

Proof. From Theorem 1 and Lemma 5, we have

$$(n+1)cV(\mathcal{K}_M) \le (n+1)V^{1/(n+1)}(\mathcal{K}_\Omega)V^{n/(n+1)}(\mathcal{K}_M),$$

which is equivalent to (8). Suppose that $M \subset \mathbb{H}^n(1/c)$. Then

$$\mathcal{K}_M = \frac{1}{c}\mathcal{K}_\Omega = \left\{\frac{1}{c}q \in \mathbb{L}^{n+1} : q \in \mathcal{K}_\Omega\right\}.$$

Thus, the equality in (8) clearly holds. Conversely, suppose that

$$V(\mathcal{K}_M) = \frac{1}{c^{n+1}} V(\mathcal{K}_\Omega). \tag{9}$$

Then, by Lemma 5, we have

$$A^{n+1}(M) \le (n+1)^{n+1} V(\mathcal{K}_\Omega) V^n(\mathcal{K}_M).$$

By Theorem 1, we have $M \subset \mathbb{H}^n(t^*)$, where $t^* = \mathrm{d}(O, M)$, the (Lorentzian) distance from the origin O to M (cf. [2,4]). By (9), we see $t^* = 1/c$ and so $M \subset \mathbb{H}^n(1/c)$. □

Theorem 7. *Let M be a properly located compact achronal spacelike hypersurface in* $\mathbb{L}^{n+1}$ *with mean curvature* $H \geq c > 0$. *Then*

$$A(M) \leq \frac{1}{c^n} A(\Omega) \tag{10}$$

with equality only when M is a subset of $\mathbb{H}^n(1/c)$.

Proof. From Theorems 1 and 6, we have

$$A(M) \leq \frac{1}{c^n}(n+1)V(\mathcal{K}_\Omega).$$

Since $\Omega \subset \mathbb{H}^n(1)$,

$$A(\Omega) = (n+1)V(\mathcal{K}_\Omega).$$

Thus, we get (10). Suppose that $M \subset \mathbb{H}^n(1/c)$. Then, the equality in (10) clearly holds. Conversely, suppose that

$$A(M) = \frac{1}{c^n} A(\Omega). \tag{11}$$

Then, by Theorem 1, we have

$$\left(\frac{1}{c^{n+1}}\right)^n V^n(\mathcal{K}_\Omega) \leq V^n(\mathcal{K}_M).$$

By Theorem 6, we have $M \subset \mathbb{H}^n(1/c)$. □

References

[1] L.J. Alías, J.A. Pastor, Constant mean curvature spacelike hypersurfaces with spherical boundary in the Lorentz–Minkowski space, J. Geom. Phys. 28 (1998) 85–93.
[2] H. Bahn, Isoperimetric inequalities and conjugate points on Lorentzian surfaces, J. Geom. 65 (1999) 31–49.
[3] H. Bahn, P. Ehrlich, A Brunn–Minkowski type theorem on the Minkowski spacetime, Can. J. Math. 51 (1999) 449–469.
[4] H. Bahn, S. Hong, Isoperimetric inequalities for sectors on the Minkowski 2-spacetime, J. Geom. 63 (1998) 17–24.
[5] J. Beem, P. Ehrlich, K. Easley, Global Lorentzian Geometry (2/e), Dekker, New York, 1996.
[6] C.-C. Hsiung, Some integral formulas for closed hypersurfaces, Math. Scand. 2 (1954) 286–294.
[7] P.L. Lions, F. Pacella, Isoperimetric inequalities for convex cones, Proc. Am. Math. Soc. 109 (1990) 477–485.
[8] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.